Recall: Second order linear homogeneous ODE w/ const coeff.

$$ay'' + by' + cy = 0, \quad a, b, c \text{ real numbers}$$

Characteristic equation: $ar^2 + br + c = 0$

Characteristic roots: r_1, r_2

Case I: r_1, r_2 real distinct.

General solution $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

Case II: $r_1 = r_2 = r$ repeated (automatically real)

General solution $y = C_1 e^{rt} + C_2 t e^{rt}$

Case III: $r_1 \neq r_2$ complex. Write $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$

General solution $y = C_1 e^{\alpha t} \cos\beta t + C_2 e^{\alpha t} \sin\beta t$

Leftovers for the previous class: repeated root case. how comes the t factor in the other solution.

Variation of Parameters

Philosophy: If you know one solution y_1, then since the ODE is linear, cy_1 will also be a solution for any number c.

$\uparrow$
parameter.

Change this numeric parameter into a function, i.e., set $y_2(t) = u(t) \cdot y_1(t)$, then plug it into the ODE to find $u(t)$.

① Application to homogeneous ODE.

Knowing y_1 is a solution to

$$y'' + p(t)y' + q(t)y = 0$$

Set $y_2(t) = u(t)y_1(t)$, put it back to the ODE.

$\Rightarrow$ another ODE with order reduced by one that leads to $u(t)$.

$$y_2'' + py_2' + qy_2 = (uy_1)'' + p(uy_1)' + q(uy_1)$$

$$(uy_1)' = u'y_1 + uy_1', \quad (uy_1)'' = u''y_1 + 2u'y_1' + uy_1''$$

$$= u''y_1 + 2u'y_1' + uy_1'' + p(u'y_1 + uy_1') + quy_1$$

$$= u''y_1 + u'(2y_1' + py_1) + u(y_1'' + py_1' + qy_1)$$

y_1 is a sol'n $\Rightarrow y_1'' + py_1' + qy_1 = 0$

$$= u''y_1 + u'(2y_1' + py_1) = 0 \quad (\text{we set } y_2 \text{ as a sol'n})$$

In other words, if $y_2 = uy_1$ is a solution, then u must satisfy

$$y_1u'' + (2y_1' + py_1)u' = 0$$

This can be regarded as a first order ODE concerning u'.

More precisely, set $v = u'$, then

$$y_1v' + (2y_1' + py_1)v = 0$$

We can solve v from this ODE $\Rightarrow u$ (by integration)

$\Rightarrow y_2$ (by multiplying u to y_1) $\Rightarrow$ Gen. sol'n: $y = C_1y_1 + C_2y_2$.

Example: $y'' - 2ry' + r^2 y = 0$. r real number.

Know from char. eqn. that $y = e^{rt}$ is a sol'n.

Set $y_2 = u(t)y_1(t) = u \cdot e^{rt}$. We know from above that u satisfies

$$y_1 u'' + (2y_1' + py_1)u' = 0$$

$$\Rightarrow \quad e^{rt}u'' + (2re^{rt} - 2re^{rt})u' = 0 \Rightarrow e^{rt}u'' = 0 \Rightarrow u'' = 0$$

Integrate: $u' = C_1$ $\qquad u' = 1$ (Set $C_1 = 1$)

Integrate again: $u = C_1 t + C_2$ $\qquad u = t$ (Set $C_2 = 0$)

$y_2 = uy_1 = (C_1 t + C_2)e^{rt} = C_1 te^{rt} + C_2 e^{rt}$ $\qquad y_2 = te^{rt}$.

This means $C_1 te^{rt} + C_2 e^{rt}$ will be another sol'n for any C_1, C_2.

When $C_1 = 0$, $C_2 = 1$, we recover y_1. $\qquad W(y_1, y_2) \not\equiv 0$

$\Rightarrow$ Gen. sol'n $y = C_1 te^{rt} + C_2 e^{rt}$. $\qquad$ Gen. sol'n $y = C_1 y_1 + C_2 y_2 = C_1 e^{rt} + C_2 te^{rt}$.

Rmks: ① Normally when solving for $u(t)$, we normally will set those arbitrary constants as concrete numbers so as to simplify the computation. However if you don't do that, then $y_2 = uy_1$ will give the general sol'n.

② When formulating the ODE concerning u, make sure your p comes from the standard form. Also notice that q is not used.

Example: $ty'' - y' - 4t^3 y = 0$. Knowing $y_1 = \sin(t^2)$ is a sol'n,

Find the general solution.

Std. form: $y'' - \frac{1}{t} y' - 4t^2 y = 0$.

Set $y_2 = u y_1$. $y_1 = \sin(t^2)$. $y_1' = 2t\cos(t^2)$ $p = -\frac{1}{t}$

$$\sin(t^2)\cdot u'' + \left(4t\cos(t^2) - \frac{1}{t}\sin t^2\right)u' = 0.$$

$$\frac{u''}{u'} = \frac{4t\cos t^2 - \frac{1}{t}\sin(t^2)}{-\sin(t^2)}$$

$$= -\frac{4t\cos(t^2)}{\sin(t^2)} + \frac{1}{t}$$

Integrate: $\ln|u'| = \ln|t| - \int \frac{4t\cos(t^2)}{\sin(t^2)}\,dt$

$$\int \frac{4t\cos(t^2)dt}{\sin t^2} \xrightarrow[du = 2t\cos(t^2)dt]{u = \sin(t^2)} \int \frac{2du}{u} = 2\ln|u| + C = 2\ln|\sin(t^2)| + C$$

$$= \ln|t| - 2\ln|\sin t^2| \qquad \text{took } C = 0$$

$$\ln|u'| = \ln\left|\frac{t}{\sin^2(t^2)}\right| \qquad \ln a - \ln b = \ln\frac{a}{b}, \quad c\ln a = \ln a^c$$

$$u' = \frac{t}{\sin^2(t^2)}$$

Integrate again: $u = \int \frac{t}{\sin^2(t^2)}\,dt$ $\qquad \frac{1}{\sin^2 t} = \csc^2 t.$

$$= \frac{-1}{2}\int \csc(t^2)\cdot d(t^2) \qquad \int \csc^2 t = -\cot t + C.$$

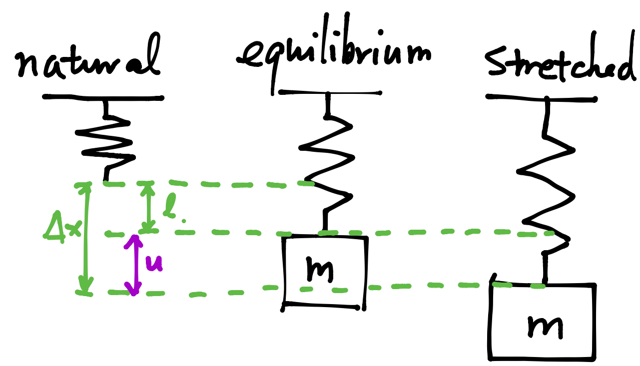
$$= \frac{-1}{2}\cot(t^2) \qquad \text{again we don't care about } C.$$

$$y_2 = u y_1 = -\frac{1}{2}\cot(t^2)\cdot\sin(t^2) = -\frac{1}{2}\cos(t^2)$$

General solution: $y = C_1 \sin(t^2) + C_2\left(-\frac{1}{2}\cos(t^2)\right)$

$$= C_1 \sin(t^2) + C_2 \cos(t^2).$$

Free Vibrations.



A mass is attached with a spring vertically. The mass is subject to:

① gravity $= mg$

② force of the spring $= k\Delta x$ (Hooke's law) Δx = displacement of the spring from the natural length.

③ Damping force $= \gamma u'$ with direction opposite to the direction of motion and being propotional to the velocity.

Let u be the displacement of the mass from the equilibrium

Take downside to be the positive direction.

$$m\frac{d^2u}{dt^2} = mg - k\Delta x - \gamma u'$$

Since $mg = k\ell$, $mg - k\Delta x = k\ell - k\Delta x = -k(\Delta x - \ell) = -ku$.

$$\Rightarrow mu'' = -ku - \gamma u' \Rightarrow mu'' + \gamma u' + ku = 0.$$

Case 1: Undamped case: $\gamma = 0$.

The ODE becomes: $mu'' + ku = 0$

Char. eqn: $mr^2 + k = 0 \Rightarrow r^2 = -\frac{k}{m} \Rightarrow r = \pm\sqrt{\frac{k}{m}}\cdot i$

Gen. sol'n: $u = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$

Write $\omega = \sqrt{\frac{k}{m}}$. Gen. sol'n $= C_1 \cos\omega t + C_2 \sin\omega t$.

With initial value specified, we can solve C_1, C_2 as concrete numbers.

$$u = C_1 \cos\omega t + C_2 \sin\omega t$$
$$= \sqrt{C_1^2 + C_2^2}\cos(\omega t - \varphi)$$

Natural frequency: $\omega = \sqrt{\frac{k}{m}}$ Period: $\frac{2\pi}{\omega}$

Amplitude: $A = \sqrt{C_1^2 + C_2^2}$

Phase: φ, determined by the angle of (C_1, C_2) on the plane.

Example: $mg = 10$ lb, $10\text{ lb} = k\cdot 2\text{ in.} = k\cdot\frac{1}{6}$ ft.

$u(0) = 2\text{ in} = \frac{1}{6}$ ft $u'(0) = -1$ ft/s.

ODE: $mu'' + ku = 0$ $m = \frac{10\text{ lb}}{32\text{ ft}/s^2} = \frac{5}{8}\text{ lb}\cdot s^2/\text{ft}$, $k = 60$ lb/ft.

$\frac{5}{8}u'' + 60u = 0 \Rightarrow u'' + 96u = 0$, $u(0) = \frac{1}{6}$. $u'(0) = -1$.

Char. roots: $r = \pm\sqrt{96}\,i = \pm 4\sqrt{6}\,i$

Gen. sol'n: $u = C_1\cos(4\sqrt{6}t) + C_2\sin(4\sqrt{6}t)$

$u(0) = \frac{1}{6} \Rightarrow C_1 = \frac{1}{6}$. $u'(0) = -1 \Rightarrow 4\sqrt{6}\,C_2 = -1 \Rightarrow C_2 = -\frac{1}{4\sqrt{6}}$.

Sol'n: $u = \frac{1}{6}\cos(4\sqrt{6}t) - \frac{1}{4\sqrt{6}}\sin(4\sqrt{6}t)$

Amplitude: $\sqrt{\frac{1}{36}+\frac{1}{96}} = \sqrt{\frac{1}{6\times 6}+\frac{1}{16\times 6}} = \sqrt{\frac{1}{12}\left(\frac{1}{3}+\frac{1}{8}\right)} = \sqrt{\frac{1}{12}\cdot\left(\frac{11}{12\times 2}\right)}$

$= \frac{1}{12}\sqrt{\frac{11}{2}}$ Stays constant $\Rightarrow$ steady oscillation.

Nat. Freq. $= 4\sqrt{6}$. Period $= \frac{2\pi}{4\sqrt{6}} = \frac{\pi}{2\sqrt{6}}$

Phase: $\varphi = \arctan\frac{6}{4\sqrt{6}} = \arctan\frac{3}{2\sqrt{6}} = \arctan\frac{\sqrt{6}}{4}$

HW. Skip 3, 4, 5. 1a.

Attendance Quiz: Knowing $y_1 = \frac{1}{t}$ is the sol'n of $t^2y'' + 3ty' + y = 0$

Find the general sol'n.

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